Fermi National Accelerator Laboratory

FERMILAB-CONF-95/379-T

Self-dual Yang-Mills Theory, Integrability and Multiparton Amplitudes

William A. Bardeen
Fermi National Accelerator Laboratory
P.O. Box 500
Batavia, IL 60510

A talk presented at the 1995 Yukawa
International Seminar, Kyoto, Japan
August 21-25, 1995

Abstract

The integrability properties of self-dual Yang-Mills theory are used to derive the structure of multiparton amplitudes in quantum chromodynamics.

Introduction.

A detailed understanding of multiparton amplitudes is crucial to the application of perturbative quantum chromodynamics to a wide variety of processes at high energy colliders. Considerable progress has been achieved in developing this understanding using a wide variety of methods. In this talk, I will review some applications of these methods and show how self-dual Yang-Mills theory may be used to develop an alternative understanding of these results and may provide an avenue for future analysis.

Structure of Multiparton Amplitudes.

Parke and Taylor [1] developed methods to analyze the structure of multiparton amplitudes based on the decomposition of QCD amplitudes using color-ordering and relations based on supersymmetry. Remarkably simple results were obtained for certain tree level amplitudes with fixed helicity structures. In this method multiparton amplitudes are decomposed as a sum over cyclic permutations of color-ordered trace of fixed helicity amplitudes

$$M_n(p_1\varepsilon_1,\cdots,p_n\varepsilon_n)=\sum_{perms} tr\left(\lambda^1\cdots\lambda^n\right)\cdot m(p_1\varepsilon_1,\cdots,p_n\varepsilon_n)$$

where $m(p_1\varepsilon_1,\cdots,p_n\varepsilon_n)$ are the gauge invariant colored-ordered amplitudes. The color-ordered amplitudes are independent within the context of a formal $1/N_C$ expansion. Supersymmetric Ward identities were used to relate multigluon amplitudes to scalar and spinor amplitudes with a simpler structure. The invariant helicity amplitudes have a simplified S-matrix structure including specific factorization properties and the isolation of soft and collinear divergences.

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

For certain helicities, the Parke-Taylor amplitudes provide remarkably simple, general relations for multigluon tree amplitudes. Expressions for these amplitudes can be written in closed form for arbitrary numbers of gluons,

$$m_n(p_1+, p_2+, \cdots, p_n+) = 0$$

$$m_n(p_1-, p_2+, \cdots, p_n+) = 0$$

$$m_n\left(p_1+, p_2+, \cdots, p_I-, \cdots, p_J-, \cdots, p_n+\right)$$
$$= i \cdot g_s^{n-2} \cdot \frac{\langle IJ\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle (n-1)n\rangle\langle n1\rangle}$$

$$\langle 12\rangle = \overline{\psi}_-(p_1)\psi_+(p_2) = \sqrt{|S_{ij}|}\cdot \exp\left(i\phi_{ij}\right)$$

Specific results were also obtained for four, five and six parton amplitudes. The method was also applied to the study of quark-gluon and quark-quark amplitudes in QCD. These methods were further developed by Mangano, Parke, Xu and others [2] and used in the study of high energy jets observed in collider experiments.

An alternative approach was developed by Berends, Giele, Kuijf and others [3] using recursion relations for multiparton amplitudes. The amplitudes with one off-shell quark or gluon could be obtained using iterative solutions of the full Yang-Mills gauge theory. Amplitudes with higher numbers of on-shell quarks or gluons could be obtained from smaller amplitudes using recursion relations based on classical solutions of the Yang-Mills field equations. The color-ordered, single off-shell multigluon amplitude may be written as

$$\left\langle A_\mu(p_n)\right\rangle_{p_1\varepsilon_1 \cdots p_{n-1}\varepsilon_{n-1}} = \sum_{perms} tr(\lambda^1 \cdots \lambda^n)\cdot J_\mu(p_1\varepsilon_1 \cdots p_{n-1}\varepsilon_{n-1})$$

$$m(1\cdots n) = \left\{i\varepsilon^\mu(p_n)\right\}\cdot p_n^2 \cdot J_\mu(1\cdots(n-1)),\ p_n^2 \to 0$$

$J_\mu(p_1\varepsilon_1 \cdots p_{n-1}\varepsilon_{n-1})$ defines the single off-shell current with n-1 gluons,

$$J_\mu(1\cdots n) = \frac{-i}{\left(p_1 + \cdots p_n\right)^2}\{\ \sum_{i=1}^{n-1} V_{3\mu}^{\nu\rho} J_\nu(1\cdots i)\cdot J_\rho((i+1)\cdots n)$$
$$+\sum_{i=1}^{n-2}\ \sum_{j=i+1}^{n-1} V_{4\mu}^{\nu\rho\tau} J_\nu(1\cdots i)\cdot J_\rho((i+1)\cdots j)\cdot J_\tau((j+1)\cdots n)\}$$

where $V_{3\mu}^{\nu\rho}, V_{4\mu}^{\nu\rho\tau}$ are the three and four point Yang-Mills vertices and $J_\mu(1) = \varepsilon_\mu(p_1)$. These recursion formulas have a particularly simple solution in the case where all of the on-shell gluons have the same helicity. In this case the fixed heliciy current has the structure,

$$J_\mu(1+,2+,\cdots,n+) = g_s^{n-1}\,\frac{\langle k-|\gamma_\mu p(1,n)|k+\rangle}{\sqrt{2}\langle k1\rangle\langle 12\rangle\cdots\langle (n-1)n\rangle\langle nk\rangle}$$

where k is a gauge defining reference momentum for the off-shell gluon. The corresponding on-shell S-matrix element, $m(1+,2+,\cdots,(n-1)+,n\pm)$, vanishes as the above current has no pole in the off-shell momentum, p_n^2. This result is consistent with the Parke-Taylor analysis. Similar expansions can be derived for single off-shell quark and anti-quark amplitudes by iterating the appropriate field equations.

The recursion relations have been used to analyze a wide variety of tree amplitudes in quantum chromodynamics. Amplitudes with maximal helicity violation are computed to all orders in the number of partons; these are just the Parke-Taylor amplitudes. The complete helicity structure has been explicitly determined for amplitudes with up to seven partons [4]. Recursion formulas can be numerically iterated to construct specific amplitudes in higher order.

Currents with two off-shell quarks or gluons are used to study the propagators of quarks and gluons in a background gauge field and are the basis for constructing on-shell amplitudes at the one-loop level. Recursion relations for these double off-shell currents can be derived from the QCD field equations [5]. The double off-shell fermion currents are given by the color-ordered sum,

$$\hat{\Psi}_{ji}(Q;1\cdots n) = g_s^n \sum_{perms}\left[T^1\cdots T^n\right]_{ji}\cdot\Psi(Q;1\cdots n)$$

and the recursion relation,

$$\Psi(Q;1\cdots n) = -\sum_{j=0}^{n-1}\Psi(Q;1\cdots j)\gamma\cdot J((j+1)\cdots n)\cdot\frac{\gamma\cdot(Q+k_1+\cdots+k_n)}{\left[Q+k_1+\cdots+k_n\right]^2}$$

with (Q, j) the antiquark and ($P=Q+k_1+\cdots+k_n$, i) the quark indices. Explicit solutions of these recursion relations have been obtained for certain fixed helicity configurations, (++⋯++) and (-++⋯++). These solutions have been used to construct specific multi-photon and multigluon fermion loop amplitudes [6]. Care must be used in regularizing the singular loop diagrams as the conventional dimensional regularization procedure does not preserve the chiral structure of the recursion relations.

Mahlon has constructed a number of multiparton one-loop amplitudes using these methods. He has explicit results for the following fermion loop amplitudes [6].

$$n\gamma,\; m(1+,2+,\cdots,n+) = 0,\; n\neq 4$$

$$m(1+,2+,3+,4+) = i\frac{e^4}{2\pi^2}\cdot\frac{\langle 12\rangle^*\langle 34\rangle^*}{\langle 12\rangle\langle 34\rangle}$$

$$n\gamma,\; m(1-,2+,\cdots,n+) = 0,\; n\neq 4$$

$$m(1-,2+,3+,4+) = i\frac{e^4}{2\pi^2}\cdot\frac{\langle 12\rangle\langle 34\rangle^*\langle 24\rangle^*}{\langle 12\rangle^*\langle 34\rangle\langle 24\rangle}$$

$e^+e^- \to n\gamma \quad (++\cdots+)$

$e^+e^- \to ng \quad (++\cdots+),\ (-+\cdots+)$

$ng,\ (++\cdots+),\ (-+\cdots+)$, (quark loop only)

An extensive program to use string methods to construct multiparton amplitudes has be carried out by Bern, Kosower, Dixon and others [7]. Using these methods, the heterotic string theory has been applied to the computation of QCD amplitudes. A particular focus of these studies has been the construction of multiparton loop amplitudes. Particular attention is paid to the regularization of the loop integrals, the unitary structure of the loop amplitudes and the soft and collinear singularities. Both quark and gluon loop amplitudes have been constructed using these methods. Complete results have been achieved for multiparton amplitudes with maximal helicity violation, n-gluons and n-photons. A full calculation of one-loop amplitudes for the 2 -> 3 processes with arbitrary helicity configurations has also been achieved [8].

Using a variety of methods from recursion relations to supersymmetry and string theory remarkable progress has been made in understanding the structure of multiparton amplitudes in Yang-Mills field theory. Many of these results are presently being applied in the analysis of multiparton processes at high energy. Another remarkable aspect of these studies is the extreme simplicity of the final expressions obtained for some of the amplitudes, particularly those processes where the partons have a fixed helicity. We will explore the possible origins of this simple structure in the following sections.

Self-dual Yang-Mills structure.

As noted in the above discussion, there is a substantial simplification for amplitudes with fixed helicity structure, (+++++++). Amplitudes become more complex as helicities are flipped (-++++++), (-++++-++++). At tree level, amplitudes with a given, fixed helicity can be generated directly from solutions of self-dual Yang-Mills theory. More complex amplitudes are generated by considering correlation functions of the self-dual theory. It is known that the self-dual theory has certain integrability properties and this structure may be exploited to provide a deeper understanding of multiparton amplitudes.

Berends and Giele [3] developed recursion formulas for gluon and spinor currents by iterating the full Yang-Mills field equations. Currents with fixed helicity are generated directly from the self-dual Yang-Mills theory. The amplitudes for one off-shell parton are given by the sum over color ordered currents,

$$\left\langle A_\mu(p_n)\right\rangle_{p_1+,p_2+,\cdots,p_{n-1}+} = \sum_{perms} tr\left(\lambda^1\cdots\lambda^n\right)\cdot J_\mu(p_1+,\cdots,p_{n-1}+)$$

where $A_\mu(p_n)$ is a solution of the self-dual field equations. The self-dual field equations are

$$\left[\sigma^{\mu},\bar{\sigma}^{\nu}\right]G_{\mu\nu}=0$$

where

$$G_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}+i\left[A_{\mu},A_{\nu}\right]$$
$$\sigma_{\mu}=(1,\vec{\sigma}),\quad \bar{\sigma}_{\mu}=(1,-\vec{\sigma})$$

It is convenient to consider the component equations,

$$G_{o+z,o-z}=G_{x+iy,x-iy},\ G_{o-z,x-iy}=0,\ G_{o+z,x+iy}=0$$

Light cone gauge can be used to write these field equations in terms of a single matrix valued scalar potential. The light cone gauge conditions are

$$A_{o}-A_{z}\equiv A_{o-z}=0$$

which imply

A $\quad G_{o+z}=-\partial_{o-z}A_{o+z}$

B $\quad G_{x+iy,x-iy}=\partial_{x+iy}A_{x-iy}-\partial_{x-iy}A_{x+iy}+i\cdot\left[A_{x+iy},A_{x-iy}\right]$

C $\quad G_{o-z,x-iy}=\partial_{o-z}A_{x-iy}=0$

D $\quad G_{o+z,x+iy}=\partial_{o+z}A_{x+iy}-\partial_{x+iy}A_{o+z}+i\cdot\left[A_{o+z},A_{x+iy}\right]$

Assuming an appropriate boundary condition, the self-duality constraints can be written as

C $\Rightarrow$ $A_{x-iy}=0$

B $\Rightarrow$ $G_{x+iy,x+iy}=-\partial_{x-iy}A_{x+iy}=G_{o+z,o-z}=-\partial_{o-z}A_{o+z}$

with solution

$$A_{x+iy}=\partial_{o-z}\Phi,\ A_{o+z}=\partial_{x-iy}\Phi$$

The equation of motion for the scalar potential, Φ, is given by

D $\Rightarrow$ $0=G_{o+z,x+iy}=\ \partial^{2}\Phi+i\cdot\left[\partial_{x-iy}\Phi,\partial_{o-z}\Phi\right]$

with

$$\bar{\sigma}\cdot A=\bar{\sigma}^{\mu}v\bar{u}\partial_{\mu}\Phi,\qquad \bar{v}=(0,1),\ \bar{u}=(1,0)$$
$$\sigma\cdot A=-v\bar{u}\sigma^{\mu}\partial_{\mu}\Phi$$

Following the analysis of Berends and Giele, an iterative solution can be obtained in terms of the color-ordered amplitude expansion for the scalar potential where all of the external states are on-shell gluons with fixed helicity,

$$\Phi = i\phi(k_1)e^{-ik_1x}T^{a_1} + \sum_n \sum_{perms} i\cdot\left[\phi(k_1)e^{-ik_1x}T^{a_1}\cdots\phi(k_n)e^{-ik_nx}T^{a_n}\right] \cdot(Q_1-Q_2)^{-1}(Q_2-Q_3)^{-1}\cdots(Q_{n-1}-Q_n)^{-1}$$

and

$$Q = k_{o+z} / k_{x-iy} = k_{x+iy} / k_{o-z}$$

With this solution the vector potential for the gluon field is given by

$$A_o = A_z = -\tfrac{1}{2}\left(\nabla_x - i\cdot\nabla_y\right)\Phi$$
$$A_x = i\cdot A_y = \tfrac{1}{2}(\partial_o + \nabla_z)\Phi$$

This solution of the self-dual field equations reproduces the solutions for the single off-shell gluon current found by Berends and Giele using the full field equations.

The above result has a remarkably simple structure which corresponds precisely with the Bethe-ansatz solutions of two dimensional integrable systems, ie. it is written as a sum over permutations of ordered plane waves multiplied by the product of the two-body S-matrices. This simple structure is related to the integrability properties of the self-dual Yang-Mills theory.

Solutions can also be obtained directly for the chiral spinor currents in self-dual gauge backgrounds. Left-handed spinors in light-cone gauge satisfy

$$\overline{\sigma}\cdot(\partial + iA)\psi(x) = 0$$

with the explicit solution for the spinor current, $\psi(x)$,

$$\psi(x) = \psi_o(k_\psi)e^{-ik_\psi x} \qquad \overline{\sigma}\cdot k_\psi \psi_o = 0$$
$$+\sum_n \sum_{perms}\left[\phi(k_1)e^{-ik_1x}T^{a_1}\cdots\phi(k_n)e^{-ik_nx}T^{a_n}\right]\psi_o(k_\psi)e^{-ik_\psi x} \cdot(-Q_1)^{-1}(Q_1-Q_2)^{-1}\cdots(Q_{n-1}-Q_n)^{-1}(-Q_\psi)^{-1}$$

Similar solutions are obtained for right-handed spinor currents in light-cone gauge,

$$\sigma\cdot(\partial+iA)\chi(x)=0$$

$$\chi(x)\equiv\left[\begin{matrix}\partial_x-i\partial_y\\ \partial_o-\partial_z\end{matrix}\right]\ \xi(x)$$

$$\xi(x)=\xi_o e^{-ik_\chi x}$$
$$+\sum_n\sum_{perms}\left[\phi(k_1)e^{-ik_1x}T^{a_1}\cdots\phi(k_n)e^{-ik_nx}T^{a_n}\right]\xi_o e^{-ik_\chi x}$$
$$(-Q_1)^{-1}(Q_1-Q_2)^{-1}\cdots(Q_n-Q_\chi)^{-1}(-Q_\chi)^{-1}$$

Both of the above spinor amplitudes reflect the Bethe-ansatz structure which follows from integrability of the combined gauge-spinor system.

Self-dual Yang-Mills theory has been extensively studied as a proto-typical integrable system Much progress has been made in understanding the structure and implications of integrable systems [9]. It is known that self-dual Yang-Mills theory generates many of the known integrable lower dimensional systems through appropriate reductions. These results should be adapted to the analysis of multiparton amplitudes directly in four dimensions. Such methods could be used to understand conservation laws and construct correlation functions of the self-dual theory.

Our solutions to the self-dual Yang-Mills theory apply directly to amplitudes involving partons with a single fixed helicity. More complex amplitudes may be generated from correlation functions computed in the self-dual background. The matrix element,

$$\left\langle A_\mu(p)\right\rangle_{p_1+,\cdots,p_n+},$$

generates the gluon currents. Amputation of these currents give the on-shell amplitudes m(+++···+++) and m(-++···+++) which vanish at tree level. The correlation function,

$$\left\langle A_\mu(p)A_\nu(p')\right\rangle_{p_1+,\cdots,p_n+},$$

generates the double off-shell currents and the on-shell amplitudes with two flipped helicities, m(-++···++-++···++). These amplitudes are known to have the simple structure of the Parke-Taylor amplitudes [1] for arbitrary numbers of positive helicity gluons. This simple structure is presumably related to the integrability properties of the self-dual theory. Can the methods developed to study integrable systems be applied to the construction of these correlation functions? It may be possible to develop a natural understanding of the more complex amplitudes from the integrable structure of the more complex correlation functions needed to construct these amplitudes. Integrability is presumably broken by loop amplitudes, but integrability may still play an important impact in determining their structure.

Anomalies may play a special role in determining the structure of some amplitudes. The on-shell amplitudes, m(+++···+++) and m(+-+···+++), vanish as a result of the conservation laws of the self-dual Yang-Mills theory. However, these amplitudes do not vanish at one-loop but have an extremely simple structure [6].

$$m(1+,2+,3+,4+) = i\frac{e^4}{2\pi^2}\cdot\frac{\langle 12\rangle^*\langle 34\rangle^*}{\langle 12\rangle\langle 34\rangle}$$

$$m(1-,2+,3+,4+) = i\frac{e^4}{2\pi^2}\cdot\frac{\langle 12\rangle\langle 34\rangle^*\langle 24\rangle^*}{\langle 12\rangle^*\langle 34\rangle\langle 24\rangle}$$

These loop amplitudes are finite and have no discontinuities. This behavior is analogous to that found for the usual chiral anomalies generated by the chiral fermion loops. This structure may result from the anomalous conservation of the currents associated with the integrability of the self-dual Yang-Mills theory.

Conclusions.

The integrability properties of self-dual Yang-Mills theory may provide a deeper understanding of the simple structure discovered for multiparton amplitudes in QCD and QED. The fundamental structure of the gluon and quark currents are exactly reproduced using the self-dual Yang-Mills formulation. The methodology of integrable systems may have important applications to the study of multiparton amplitudes, both at tree level and at higher loop order. The approach described here may provide a complementary view to the results obtained from the application of string theory and supersymmetry to parton amplitudes.

References.

[1] S. Parke and T. Taylor, Nucl. Phys. B269, 410(1986); Phys. Rev. Lett. 56, 2459(1986); Phys. Rev. D35, 313(1987).

[2] M. Mangano and S. Parke, Nucl. Phys. B299, 673(1988); Phys. Rev. D39, 758(1989); Phys. Rep. 200, 301(1991); M. Mangano, S. Parke and Z. Xu, Nucl. Phys. B298, 653(1988); B299, 673(1988).

[3] F. Berends and W. Giele, Nucl. Phys. B294, 700(1987); Nucl. Phys. B306, 759(1988); F. Berends, W. Giele and H. Kuijf, Nucl. Phys. B321, 39(1989); Phys. Lett. B232, 266(1989); Nucl. Phys. B333, 120(1990).

[4] F. Berends and H. Kuijf, Nucl. Phys. B353, 59(1991); F. Berends, W. Giele, H. Kuijf, Nucl. Phys. B333, 120(1990).

[5] F. Berends and W. Giele, Nucl. Phys. B306, 759(1989), Mahlon, Phys. Rev. D49, 4438(1994)

[6] G. Mahlon, T.-M. Yan, Phys. Rev. D47, 1776(1993); G. Mahlon, Phys. Rev. D49, 2197(1994); Phys. Rev. D49, 4438(1994).

[7] Z. Bern and D. Kosower, (1987-1995); D. Kosower, V. Nair, and B. Lee, (1988); Z. Bern and D. Kosower, Nucl. Phys. B379, 451(1992); Z. Bern, L. Dixon, D. Kosower, Phys. Rev. Lett. 70, 2677(1993); Z. Bern, L. Dixon and D. Dunbar, Nucl. Phys. B435, 435(1995).

[8] $ggggg$: Z. Bern, L. Dixon, D. Kosower, Phys. Rev. Lett. 70, 2677(1993); $q\bar{q}q\bar{q}g$: Z. Kunszt, A. Signor, Z. Trócsányi, Phys. Lett. B336, 529(1994); $q\bar{q}ggg$: Z. Bern, L. Dixon, D. Kosower, Nucl. Phys. B437, 259(1995).

[9] C.-N. Yang, Phys. Rev. Lett. 38, 1377(1977), A. Belavin and V. Zakharov, Phys. Lett. 73B, 53(1978); C.-H. Tze and Y.-S. Wu, Nucl. Phys. B204, 118(1982); E. Sklyanin, (1982) and lectures published in "Quantum Group and Quantum Integrable Systems", Mo-Lin Ge, ed., World Scientific, Singapore (1992); L.-L. Chau, Phys. Rev. D26, 3581(1982); J. Schiff, Preprint IASSNS-HEP-92/34 (1992); L.-L. Chau and I. Yamanaka, (1991-1993); L. Faddeev, Preprint ITP-SB-94-11 (1994); C. Zachos and T. Curtright, Phys. Rev. D49, 5408(1994); many others.